Projecting Massive Scalar Fields to Null Infinity

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Abstract

It is known that, in an asymptotically flat spacetime, null infinity cannot act as an initial-value surface for massive real scalar fields. Exploiting tools proper of harmonic analysis on hyperboloids and global norm estimates for the wave operator, we show that it is possible to circumvent such obstruction at least in Minkowski spacetime. Hence we project norm-finite solutions of the Klein-Gordon equation of motion in data on null infinity and, eventually, we interpret them in terms of boundary free field theory.

1   Introduction.

In the study of classical fields over four dimensional Lorentzian curved backgrounds, Penrose conformal completion techniques have played since their introduction a pivotal role.

In particular the related notion of asymptotic simplicity/flatness entails the embedding of a (four dimensional) physical spacetime $(M^4, g_{\mu\nu})$ as a bounded open set in an unphysical background $(\widehat{M}^4, \widehat{g}_{\mu\nu})$ being $\widehat{g}$ a conformal rescaling of g. In this setting the image of M^4 in $\widehat{M}^4$ can be naturally endowed with a boundary structure usually referred to as $\Im^{\pm}$ *i.e.* future or past null infinity.

Heuristically the endpoint of all the null geodesics in $(M^4, g_{\mu\nu})$, $\Im^{\pm}$ is thus the geometrical locus where the trajectory of zero rest mass particles end. Hence it is manifest how null conformal boundaries can be exploited as a powerful tool to study either the asymptotic properties of radiation fields associated to massless wave equations, either the scattering properties of massless fields [MaNi04].

Furthermore, from the perspective of quantum field theory over curved backgrounds, $\Im^{\pm}$ plays a key role in the realization of the holographic principle. The latter conjectures that the information of any field theory on a D-dimensional Lorentzian background M can be recovered by means of a suitable second field theory constructed over a codimension one submanifold Σ embedded in M. Hence, in asymptotically flat spacetimes it is natural to conjecture that the role of Σ is played by the null conformal boundary and this idea has been successfully investigated both at a classical and at a quantum level in [DMP06].

To better understand the main rationale underlying the success of Penrose conformal techniques from a field theoretical perspective, let us consider a working example, namely the massless Klein-Gordon real scalar field ψ conformally coupled to gravity in a globally hyperbolic and asymptotically flat spacetime M^4. Barring a few technical assumptions, each solution of $\left[\Box_g - \frac{R}{6}\right]\psi = 0$ with compactly supported initial data on a Cauchy surface can be mapped into a solution of

$$\widehat{\Box}_{\widehat{g}}\widetilde{\psi} - \frac{\widehat{R}}{6}\widetilde{\psi} = 0, \tag{1}$$

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where $\widetilde{\psi} \doteq \Omega^{-1}\psi$. Although $\widetilde{\psi}$ is strictly defined only over the image of M^4 in $\widehat{M}^4$, global hyperbolicity of $\widehat{M}^4$ and uniqueness of solutions for second order hyperbolic PDE, allows us to extend $\widetilde{\psi}$ to a smooth solution for (1) over all $\widehat{M}^4$. Accordingly we can define the projection of $\widetilde{\psi}$ over the boundary $\Im^{\pm}$ simply as its restriction: $\Psi_{\pm} \doteq \widetilde{\psi}|_{\Im^{\pm}} \in C^{\infty}(\Im^{\pm})$. It is Ψ the key ingredient to study properties of bulk physical phenomena starting from boundary data in the unphysical spacetime as exploited, to quote just a few examples, in [DMP06, MaNi04].

Nonetheless the situation is not heavenly as it may seem since the above construction drastically fails whenever one considers massive fields. Even in the simplest situation of the Klein-Gordon scalar field on flat Minkowski spacetime, conformal invariance of the equation of motion is broken. Furthermore it has been argued in [Hel93, Win88] that $\Im^{\pm}$ cannot be used as an initial value surface for massive fields and that it is not possible to project any solution of $\left[\Box_g - m^2\right]\psi = 0$ into a smooth function over $\Im^{\pm}$. This result has been established with an elegant argument in [Hel93]: the space of sections of any vector bundle on $\Im^{\pm}$ which is homogeneous for the action of the Poincaré group carries only massless representations[1].

Hence it seems impossible to exploit the powerful means of Penrose compactification whenever we deal with solutions of partial differential equations containing a term proportional to a scale length such as the mass. In other words, since the information of the data evolving to infinity along causal timelike curves flows in the unphysical spacetime $\widehat{M}^4$ to future timelike infinity i^+ (a codimension 2 submanifold of $\widehat{M}^4$ hence not a proper boundary), it seems impossible to exploit null infinity as a tool to study massive fields.

The aim of this paper is to provide a way to circumvent the above obstruction at least in Minkowski background. In particular we will exploit both tools of harmonic analysis and global norm estimates for the wave equation in order to project a solution for the massive Klein-Gordon equation of motion into meaningful data over null infinity.

More in detail, the outline of the analysis and hence of the paper will be the following: in the next subsection we recollect some basic details about the notion of asymptotic flatness. In section 2, instead, we specialise to Minkowski background and we consider solutions of the massive Klein-Gordon equation of motion satisfying a finite norm condition in such a way that their Fourier transform is a square integrable function over the mass hyperboloid $\mathbb{H}_m$. Exploiting a few results due to Strichartz on harmonic analysis over hyperboloids we shall introduce a unitary map between two copies of $L^2(\mathbb{H}_m)$ and the space of square integrable function over the light cone C. Furthermore such a map will also act as an intertwiner between the quasi-regular representations of the Lorentz group on $L^2(\mathbb{H}_m)$ and $L^2(\mathcal{C})$.

Afterward, as a first step, we exploit global norm estimates to associate to each square integrable function over the light cone a norm finite solution for the wave equation in Minkowski spacetime. By means of Penrose compactification techniques and trace theorems, we project

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[1]A reader familiar with Penrose compactification techniques could argue that the relevant symmetry group on null infinity is not the Poincaré but the BMS group which is the semidirect product between the proper ortochronous component of the Lorentz group and the smooth functions over the 2-sphere thought as an Abelian group under addition. Nonetheless, since in Helfer construction, the key role is played by the translational subgroup of the Poincarè group, the result can be extended also in a BMS framework remembering that it exists a four dimensional normal subgroup of the full BMS group homeomorphic to T^4 [DMP06].

these functions on null infinity.

Eventually, in section 3, we show how the projected data can be interpreted in terms of a diffeomorphism invariant field theory intrinsically constructed over null infinity.

1.1. *On asymptotically flat spacetimes.* In this section we recollect some known facts about the definition and the properties of asymptotically flat spacetime. Although we are going to work in Minkowski background, the following summary can be useful for a twofold reason: from one side in section 3 we shall interpret the projection of the data from a bulk massive scalar field in terms of a field theory on future null infinity whereas, from the other side, we look at this paper as the first step to solve the same problem on a generic asymptotically flat spacetime. Hence it could be interesting to understand where our construction relies on properties specific of Minkowski spacetime and where, on the opposite, out results could be traded to a more general scenario.

In the literature there are several different notions of asymptotic flatness at (future or past) null infinity which are obviously all equivalent if the bulk spacetime is Minkowski; hence a reader familiar to any of these can skip to next section without a second thought. We shall instead adopt the specific definition first introduced by Friedrich (see [Fri88] and references therein from the same author) of a class of spacetimes which are flat at future null infinity and they admit future time completion at i^+. The reason for this choice lies in the realm of quantum field theory of curved background. In particular in [DMP06] it has been shown that it is possible to project the Weyl *-algebra of observables for a real massless scalar field in Minkowski spacetime as a subsector of a suitable counterpart at null infinity because the Lichnerowicz propagator for the wave operator is strictly supported on the light cone. On the opposite, in a generic curved background, a priori this does not held true since the support includes a tail strictly contained in the cone and, hence, in the conformal completion language propagating at future timelike infinity. Thus in order to recast the result of [DMP06] in a generic scenario Friedrich definition is the most appealing (to this avail see the analysis in [Mor06]).

In detail a four dimensional future time oriented spacetime M^4 with a smooth metric $g_{\mu\nu}$ which solves the vacuum Einstein equation is called an *asymptotically flat spacetime with future time infinity* i^+ if it exists a second four dimensional spacetime $(\widehat{M}, \widehat{g}_{\mu\nu})$ with a preferred point i^+, a diffeomorphism $\lambda : M \to \lambda(M) \subset \widehat{M}$ and a non negative scalar function Ω on $\lambda(M)$ such that $\widehat{g} = \Omega^2 \lambda^* g$ and the following facts hold:

1. $J^-(i^+; \widehat{M})$ is closed and $\lambda(M) = J^-(i^+) \setminus \partial J^-(i^+; \widehat{M})$. Moreover $\partial \lambda(M) = \Im^+ \cup i^+$ where $\Im^+ \doteq \partial J^-(i^+; \widehat{M}) \setminus \{i^+\}$ is future null infinity.

2. $\lambda(M)$ is strongly causal.

3. Ω can be extended to a smooth function on $\widehat{M}$.

4. $\Omega|_{\partial J^-(i^+; \widehat{M})} = 0$, but $d\Omega(x) = 0$ for $x \in \Im^+$ and $d\Omega(i^+) = 0$, but $\widehat{\nabla}_\mu \widehat{\nabla}_\nu \Omega(i^+) = -2\widehat{g}_{\mu\nu}(i+)$.

5. If $n^\mu \doteq \widehat{g}^{\mu\nu} \widehat{\nabla}_\nu \Omega$ then it exists a strictly positive smooth function ω, defined in a neighbourhood of $\Im^+$ and satisfying $\widehat{\nabla}_\mu(\omega^4 n^\mu) = 0$ on $\Im^+$, such that the integral curves of $\omega^{-1} n^\mu$ are complete on $\Im^+$.

From now we shall refer to $\lambda(M)$ simply as M since no confusion will arise in the manuscript due to this identification. Furthermore we point out that, with minor adaption, the above definition can be recast for spacetimes which are asymptotically flat with past time infinity i_- and henceforth we shall refer only to $\Im^+$ though the reader is warned that all our results hold identically for $\Im^-$.

Thus let us consider any asymptotically flat spacetime as per the previous definition; the metric structure of future null infinity is not uniquely determined but it is affected by a gauge freedom in the choice of the compactification factor namely, if we rescaled Ω as $\omega\Omega$ with $\omega \in C^\infty(\Im^+, \mathbb{R}^+)$, the topology and the differentiable structure of future null infinity is left unchanged. Hence the difference between the possible geometries for the conformal boundary is caught by equivalence classes of the following triplet of data $(\Im^+, n_a, h^{ab})$ where $\Im^+$ stands for the $S^2 \times \mathbb{R}$ topology of null infinity, $n_a \doteq \widehat{\nabla}^a \Omega$ (being $\widehat{\nabla}$ the covariant derivative with respect to $\widehat{g}_{ab}$) and $h_{ab} \doteq \widehat{g}_{ab}|_{\Im^+}$. Two triplets $(\Im^+, n_a, h^{ab})$ and $(\Im^+, n'_a, h'^{ab})$ are called equivalent iff it exists a gauge factor ω such that $h'_{ab} = \omega^2 h_{ab}$ whereas $n'^a = \omega^{-1} n^a$.

The set of all these equivalence classes is *universal* in the sense that, given any two asymptotically flat spacetime M_1 and M_2 with associated triplets $(\Im_1^+, n_{1a}, h_1^{ab})$ and $(\Im_2^+, n_{2a}, h_2^{ab})$, it always exists a diffeomorphism $\gamma \in Diff(\Im_1^+, \Im_2^+)$ such that $\gamma^* h_2^{ab} = h_1^{ab}$ and $\gamma_* n_{1a} = n_{2a}$.

The set of all group elements $\gamma \in Diff(\Im^+, \Im^+)$ mapping a triplet into a gauge equivalent one[2] is called the Bondi-Metzner-Sachs group (BMS). It is always possible to choose ω in such a way that on null infinity we can introduce the so-called Bondi frame $(u, z, \bar{z})$ where u is the affine parameter along the null complete geodesics generating $\Im^+$ and $(z, \bar{z})$ are the complex coordinates construct out of a stereographic projection from $(\theta, \varphi) \in \mathbb{S}^2$, then the BMS group is $SO(3,1) \ltimes C^\infty(\mathbb{S}^2)$ acting as

$$u \longrightarrow u' = K_\Lambda(z, \bar{z})\left(u + \alpha(z, \bar{z})\right), \tag{2}$$

$$z \longrightarrow \Lambda z \doteq \frac{az+b}{cz+d}, \quad \bar{z} \longrightarrow \Lambda \bar{z} \doteq \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}, \tag{3}$$

where Λ is identified with the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ whereas

$$K_\Lambda(z, \bar{z}) = \frac{1+|z|^2}{|az+b|^2 + |cz+d|^2}.$$

A direct inspection of this formula shows that the BMS group is a regular semidirect product and it is much larger than the Poincaré group. In a generic scenario such a problem cannot be easily overcome though one can recognise that any element in the Abelian ideal $C^\infty(\mathbb{S}^2)$ can be expanded in real spherical harmonics as

$$\alpha(z, \bar{z}) = \sum_{l=0}^{1} \sum_{m=-l}^{l} \alpha_{lm} S_{lm}(z, \bar{z}) + \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \alpha_{lm} S_{lm}(z, \bar{z}). \quad \forall \alpha(z, \bar{z}) \in C^\infty(\mathbb{S}^2)$$

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[2] Although at first sight we are considering a subgroup of the whole set of diffeomorphism, one should take into account that the constraint we impose is equivalent to require that the bulk geometry is left unchanged *i.e.* we are working on a fixed background.

Here we have separated the set of first four components - known as the translational component of the BMS group - since it is homeomorphic to the Abelian group T^4. Furthermore the following proposition holds:

Proposition 1.1. *The subset $SO(3,1) \ltimes T^4$ of the BMS group made of elements $(\Lambda, \alpha(z, \bar{z}))$, where $\alpha(z, \bar{z})$ is a real linear combinations of the first four spherical harmonics, is a BMS subgroup and if we associate to $\alpha(z, \bar{z})$ the vector*

$$a^\mu = -\sqrt{\frac{3}{4\pi}} \left(\frac{a_{00}}{\sqrt{3}}, a_{1-1}, a_{10}, a_{11} \right),$$

the action of $\Lambda \in SO(3,1)$ on a^μ is equivalent to the transformation of the 4-vector in Minkowski background under the standard Lorentz action.

The proof of this theorem has been given in propositions 3.11 and 3.12 in [DMP06].

On the opposite we wish to underline that, in a generic asymptotically flat spacetime, we cannot exploit this last statement to select a preferred Poincaré subgroup in the BMS since, acting per conjugation over the above $SO(3,1) \ltimes T^4$ subset with any element $(I, S_{lm}(z, \bar{z})) \in SO(3,1) \ltimes C^\infty(\mathbb{S}^2)$ with $l > 1$ we end up with a different though equivalent Poincaré subgroup. Nonetheless, since in this paper we are taking into account only Minkowski background, we can exploit a result due to Geroch, Ashtekar and Xanthopoulos [AsXa78, Ger77] namely

Proposition 1.2. *In any asymptotically flat spacetime $(M, g_{\mu\nu})$ it holds*

a) *any Killing vector ξ in M smoothly extends to a Killing vector $\widehat{\xi}$ in $\widehat{M}$ and the restriction $\tilde{\xi}$ of the latter to $\Im$ is tangent to null infinity, it is uniquely determined by ξ and it generates a one-parameter subgroup of the BMS.*

b) *the map $\xi \to \tilde{\xi}$ is injective and, if the one-parameter subgroup of the BMS generated by $\tilde{\xi}$ lies in $C^\infty(\mathbb{S}^2)$ then it must also be a subgroup of T^4.*

According to the last proposition, in a Minkowski background, the Poincaré isometries identify a preferred subgroup of the BMS group *i.e.* the set

$$\mathcal{R} = \left\{ (\Lambda, \alpha(z, \bar{z})) \mid \alpha(z, \bar{z}) = a^0 + a^1 \frac{z + \bar{z}}{1 + |z|^2} + a^2 \frac{z - \bar{z}}{1 + |z|^2} + a^3 \frac{|z|^2 - 1}{1 + |z|^2} \right\}, \tag{4}$$

which is homomorphic to $SO(3,1) \ltimes T^4$.

2 From Massive to Massless Scalar fields on Minkowski space-time

Let us consider four dimensional flat Minkowski spacetime $(M^4, \eta_{\mu\nu})$ and a real scalar field $\phi : M^4 \to \mathbb{R}$ satisfying the Klein-Gordon equation with squared mass $m^2 > 0$:

$$\Box_\eta \phi - m^2 \phi = 0. \tag{5}$$

In the most general framework we should seek for tempered distributions solutions to such PDE and their Fourier transform is a function supported on the mass hyperboloids $\mathbb{H}_m$ (see section IX.9 of [ReSi75]) $\eta^{\mu\nu} p_\mu p_\nu = p_0^2 - \sum_{i=1}^3 p_i^2 = m^2$, being $p_\mu = (p_0, p_i)$ with $i = 1, .., 3$ the standard global coordinates[3] on each fibre of the cotangent bundle T^*M^4 canonically identified as $\mathbb{R}^4 \times \mathbb{R}^4$.

The mass hyperboloids can be parameterised with the coordinates $r \doteq |\vec{p}| \equiv \left(\sum_{i=1}^3 p_i^2 \right)^{\frac{1}{2}} \in [0, \infty)$, $\vec{\zeta} \doteq \frac{\vec{p}}{|\vec{p}|} \in \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and $\epsilon \doteq \frac{p_0}{|p_0|} = \pm 1$. The variable ϵ provides a way to distinguish in $\mathbb{R}^4$ between the upper and lower hyperboloid and we will keep track of it for the sake of generality. An interested reader can adapt the following constructions to a single hyperboloid with minor efforts.

Furthermore, identifying $\mathbb{H}_m$ with the coset $\frac{O(3,1)}{O(3)}$, we can endow it with the $O(3,1)$ invariant measure $d\mu(\mathbb{H}_m) = \frac{r^2}{\sqrt{r^2+m^2}} dr d\zeta$. Hence we can take into account only the solutions of (5) that are finite with respect to a suitable norm $i.e.$, following the conventions of [Stri71], it must exists a real number $\alpha \geq 0$ and a function $f(r, \zeta, \epsilon)$ such that, being $\vec{x}$ the spatial component of x^μ and $\cdot$ the standard Euclidean scalar product on $\mathbb{R}^3$,

$$\phi(x^\mu) = \sum_{\epsilon = \pm 1} \int_{\mathbb{H}_m} d\mu(\mathbb{H}_m) \, e^{ir\vec{x}\cdot\vec{\zeta}} e^{-i\sqrt{r^2+m^2}t\epsilon} f(r, \zeta, \epsilon)$$

and

$$\|\phi\|_\alpha^2 = \sum_{\epsilon = \pm 1} \int_{S^2} d\zeta \int_0^\infty dr \left| (r^2 + m^2)^{\frac{\alpha}{2}} f(r, \zeta, \epsilon) \right|^2 d\mu(\mathbb{H}_m) < \infty. \tag{6}$$

Dropping from now any on all references to $d\mu(\mathbb{H}_m)$, we shall call the Hilbert spaces of functions satisfying (6) as $L^2_\alpha(\mathbb{H}_m)$ and, out of a direct inspection of the above formula, the following chain of inclusions holds: $L^2(\mathbb{H}_m) \equiv L^2_0(\mathbb{H}_m) \subset L^2_\alpha(\mathbb{H}_m) \subset L^2_{\alpha'}(\mathbb{H}_m)$ for all $0 < \alpha < \alpha'$.

To summarise the key point, the constraint (6) allows us a way to select only those solutions ϕ of (5) whose Fourier transform f is at least square integrable on the mass hyperboloid with

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[3]The symbols are here adopted with respect to the standard high energy physics terminology though we do not seek at the moment any physical interpretation of the forthcoming analysis leaving it for the conclusions.

respect to the $O(3, 1)$-invariant measure. Furthermore we can require the $O(3, 1)$ group to act on f with the quasi-regular scalar representation *i.e.* for any $\Lambda \in O(3, 1)$ and for any $p_\mu \in \mathbb{H}_m \hookrightarrow \mathbb{R}^4$

$$U(\Lambda)f(p_\mu) = f(\Lambda^{-1}p_\mu), \quad f \in L^2(\mathbb{H}_m,) \tag{7}$$

being U unitary strongly continuous but not irreducible.

Henceforth our plan is to discuss and later to exploit the following Strichartz result: it is possible to construct an operator T from $L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$ into the space of square-integrable functions over the light cone with respect to the $O(3, 1)$-invariant measure and T is also a unitary intertwiner[4] between the quasi-regular $O(3, 1)$-representations.

2.1. *From hyperboloids to light cones.*

The analysis and the statements in this section are based upon the theorems proved in [Stri73] even though part of the results have been independently developed also in [LNR67] and, by means of integral transforms associated to horospheres. The proof of most of the following results strongly relies upon the embedding of the mass hyperboloid and of the light cone in $\mathbb{R}^4$. All the analysis can be recast in terms of the intrinsic structures over these symmetric space and we refer to [Ros78] for an interested reader.

As a starting point we shall briefly discuss and characterise some properties of square integrable functions over the light cone. Let us quickly recall that the latter is the geometric locus $\mathcal{C} = \left\{ p_\mu = (p_0, p_i), \ \eta^{\mu\nu}p_\mu p_\nu = p_0^2 - |\vec{p}\,|^2 = 0 \right\} \setminus (0, 0)$ where $p_\mu = (p_0, \vec{p}) = (p_0, p_i)$ with $i = 1, ..., 3$ are the same global coordinates introduced in the previous section. As for $\mathbb{H}_m$ we can set a more convenient coordinate system and an $O(3, 1)$-invariant measure which are basically constructed with a limiting procedure (*i.e.* $m \to 0$) from the counterpart on the mass hyperboloid. Namely, if we refer to $r \doteq |\vec{p}\,| \in (0, \infty)$, $\vec{\zeta} \doteq \frac{\vec{p}}{|\vec{p}|} \in \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and $\epsilon \doteq \frac{p_0}{|p_0|} = \pm 1$, the measure is $d\mu(\mathcal{C}) = r dr d\zeta$. Here the two values of ϵ allow us to distinguish between the future and the past light cone and, as for the massive case, we keep track of them for the sake of completeness.

The next step consists of a specific characterisation for square integrable functions over the light cone with respect to $d\mu(\mathcal{C})$. Let us consider the set $\mathcal{D}^0_\sigma$ and $\mathcal{D}^1_\sigma$ respectively as even and odd smooth functions over $\mathcal{C}$ homogeneous of degree σ in the r-variable *i.e.* of the form $r^\sigma g(\zeta, \epsilon)$. Then the following proposition holds

Proposition 2.1. *If $\sigma = -1 + i\rho$ with $\rho \in \mathbb{R}$, then $\mathcal{D}^0_\sigma$ and $\mathcal{D}^1_\sigma$ can be closed to Hilbert space $\mathcal{H}^0_\sigma$ and $\mathcal{H}^1_\sigma$ with respect to the norm*

$$||r^\sigma g(\zeta, \epsilon)||^2_\sigma = \sum_{\epsilon = \pm 1} \int_{\mathbb{S}^2} d\zeta |g(\zeta, \epsilon)|^2.$$

Furthermore

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[4]We recall that, given a group G with the representations U and U' on the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, a bounded linear map $T : \mathcal{H} \to \mathcal{H}'$ is called an intertwiner if $U'(g)T = TU(g)$ for all $g \in G$.

1. the quasi-regular $O(3,1)$ scalar representation acting on the functions over $\mathcal{C}$ as

$$U'(\Lambda)F(p_\mu) = F(\Lambda^{-1}p_\mu), \quad \forall p_\mu \in \mathcal{C} \hookrightarrow \mathbb{R}^4 \wedge \forall F \in L^2(\mathcal{C}),$$

is strongly continuous unitary and irreducible on both $\mathcal{H}_\sigma^0$ and $\mathcal{H}_\sigma^1$.

2. for any $F \in L^2(\mathcal{C})$ it exists a unique function φ_0 in $\mathcal{H}_\sigma^0$ and φ_1 in $\mathcal{H}_\sigma^1$ such that, calling $F_0(p_\mu) = \frac{1}{2}\left(F(p_\mu) + F(-p_\mu)\right)$ and $F_1(p_\mu) = \frac{1}{2}\left(F(p_\mu) - F(-p_\mu)\right)$, then

$$||F_j(p_\mu)||_{L^2(\mathcal{C})} = \sum_{\epsilon=\pm 1} \int_{-\infty}^{\infty} \frac{d\rho}{2\pi} \int_{\mathbb{S}^2} d\zeta \, |\varphi_j(\rho,\zeta,\epsilon)|^2, \quad j=0,1 \tag{8}$$

and

$$F_j(r,\zeta,\epsilon) = \sum_{\epsilon=\pm 1} \int_{-\infty}^{\infty} \frac{d\rho}{2\pi} r^{-1+i\rho} \varphi_j(\rho,\zeta,\epsilon). \quad j=0,1 \tag{9}$$

The image of the map $F \to (\varphi_0,\varphi_1)$ is onto all pairs with a finite right hand side in (8).

Proof. We here sketch the main details of the proof as in [Stri73]. To start, let us notice that the norm over $\mathcal{D}_\sigma^0$ and $\mathcal{D}_\sigma^1$ is well defined since, up to the sum over ϵ, it is equivalent to the norm over $L^2(\mathbb{S}^2, d^2x)$ being d^2x the Lesbegue measure on $\mathbb{S}^2$.

The unitarity and strong continuity of the quasi-regular representation arises due to the $O(3,1)$-invariance of the measure on the light cone. Hence for any $F(p_\mu) \in L^2(\mathcal{C})$ with $p_\mu \in \mathbb{R}^4$ satisfying $\eta^{\mu\nu}p_\mu p_\nu = 0$, it holds:

$$\int_{\mathcal{C}} d\mu(\mathcal{C})|U'(\Lambda)F(p_\mu)|^2 = \int_{\mathcal{C}} d\mu(\mathcal{C})|F(\Lambda^{-1}p_\mu)|^2 = \int_{\mathcal{C}} d\mu(\Lambda\mathcal{C})|F(p_\mu)|^2 = \int_{\mathcal{C}} d\mu(\mathcal{C})|F(p_\mu)|^2,$$

where, in the second equality, we performed the coordinate change $p_\mu \to \Lambda p_\mu$.

To prove irreducibility let us note that any function $f \in \mathcal{H}_\sigma^j$ with $j=0,1$ can be decomposed in spherical harmonics *i.e.* $f(r,\zeta,\epsilon) = r^\sigma g(\zeta,\epsilon) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\zeta)\epsilon^k r^\sigma$ where $k=0,1$ and the coefficients a_{jm} must vanish if $j=0$ and $l+k$ is odd or if $j=1$ and $l+k$ is even. Consider now, as a special case, a function in $\mathcal{H}_\sigma^j$ with all but one of the coefficients a_{lm} equal to zero. We show now that the action of the quasi-regular $O(3,1)$ representation generates a second function with the coefficients $a_{l+1,m} \neq 0$. To this avail let us choose an element of $SU(1,1) \subset O(3,1)$ parameterised by an angle α, apply it to f and then let us differentiate with respect to α. The resulting function f' evaluated in $\alpha=0$ is

$$f'(r,\zeta,\epsilon) = \frac{(\sigma-l)(l+1)}{1+2l} r^\sigma Y_{l+1}(\zeta)\epsilon^{k+1}.$$

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Since all these operations should map any irreducible subspace of $\mathcal{H}_\sigma^j$ into itself, the statement in point 1. of the theorem holds.

To demonstrate point 2. let us associate to any $F(p_\mu) \equiv F(r, \zeta, \epsilon) \in L^2(\mathbb{C})$, the functions $g_j(r, \zeta, \epsilon) = rF_j(r, \zeta, \epsilon)$ with $j = 0, 1$. Hence for each j

$$\int_\mathbb{C} d\mu(\mathbb{C})|F_j(r, \zeta, \epsilon)|^2 = \int_{\mathbb{S}^2} d\zeta \int_0^\infty dr \ r^{-1}|g_j(r, \zeta, \epsilon)|^2 < \infty,$$

which implies that $\int_0^\infty dr r^{-1}|g_j(r, \zeta, \epsilon)|^2 < \infty$ per Fubini's theorem. Hence we can apply Mellin inversion theorem to write $g_j(r, \zeta, \epsilon) = \int_{-\infty}^\infty d\rho \ r^{i\rho}\varphi_j(\rho, \zeta, \epsilon) = \int_{-\infty}^\infty d\rho \ e^{i\rho \ln(r)}\varphi_j(\rho, \zeta, \epsilon)$. The last identity suggests us to apply Plancherel theorem to conclude that $\int_0^\infty d\ln(r)|g_j(r, \zeta, \epsilon)|^2 = \int_{-\infty}^\infty d\rho (2\pi)^{-1}|\varphi_j(\rho, \zeta, \epsilon)|^2$ and that $\varphi_j(\rho, \zeta, \epsilon) = \int_0^\infty d\ln(r)e^{is\ln(r)}g_j(r, \zeta, \epsilon)$. Hence, upon integration over the compact S^2-coordinates we recover (8) and (9). The overall construction relies only on Mellin inversion formula and the Plancherel theorem; hence the map from F_j onto φ_j exists whenever the latter is square-integrable; this concludes the demonstration. $\qquad\square$

To conclude the analysis on the functions over a light cone, let us recall the following result still from [Stri73]:

Lemma 2.1. *Whenever $\rho \neq 0$ then*

$$A_0(\rho)\varphi(\zeta', \epsilon') = \frac{\rho}{\pi}\sum_{\epsilon=\pm 1}\int_{\mathbb{S}^2}|\vec{\zeta} \cdot \vec{\zeta}' - \epsilon\epsilon'|^{-1-i\rho}\varphi(\zeta, \epsilon)d\zeta, \tag{10}$$

$$A_1(\rho)\varphi(\zeta', \epsilon') = \frac{\rho}{\pi}\sum_{\epsilon=\pm 1}\int_{\mathbb{S}^2}|\vec{\zeta} \cdot \vec{\zeta}' - \epsilon\epsilon'|^{-1-i\rho}sgn\left(\vec{\zeta} \cdot \vec{\zeta}' - \epsilon\epsilon'\right)\varphi(\zeta, \epsilon)d\zeta, \tag{11}$$

are unitary operators respectively on odd and on even functions in $L^2(\mathbb{S}^2 \times \pm 1)$. In (10) and (11) "$\cdot$" stands for the standard Euclidean scalar product on $\mathbb{R}^3$, whereas a function $f(\zeta, \epsilon) \in L^2(\mathbb{S}^2 \times \pm 1)$ iff

$$\sum_{\epsilon=\pm 1}\int_{\mathbb{S}^2}|f(\zeta, \epsilon)|^2 < \infty.$$

We can now put together the previous lemma and proposition 2.1 in order to represent any function $F \in L^2(\mathbb{C})$ as

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$$F(p_\mu) = \frac{1}{2\pi^3} \sum_{\epsilon'=\pm 1} \int\limits_{-\infty}^{\infty} d\rho \, \rho^2 \int\limits_{S^2} d\zeta' |\vec{p} \cdot \vec{\zeta'} - p_0 \epsilon'|^{-1+i\rho} \left[\psi_0(\rho, \zeta', \epsilon') + \right.$$

$$\left. + \psi_1(\rho, \zeta', \epsilon') sgn(\vec{p} \cdot \vec{\zeta'} - p_0 \epsilon') \right], \tag{12}$$

being $\psi_k(\rho, \zeta', \epsilon') \doteq \frac{\pi}{\rho} A_k(\rho) \varphi_k(\rho, \zeta', \epsilon')$ with $k = 0, 1$ and φ_k chosen according to (9).

Let us now move back to the square integrable functions over $\mathbb{H}_m$ and, to fix notations, let us call $\widetilde{\Box} = -\frac{\partial^2}{\partial p_0^2} + \sum_{i=1}^{3} \frac{\partial^2}{\partial p_i^2}$. Switching from the coordinates (p_0, p_i) to (m, r, ζ) as introduced at the beginning of this section, the D'Alambert operator becomes $\widetilde{\Box} = -\frac{\partial^2}{\partial m^2} - \frac{3}{m} \frac{\partial}{\partial m} + \frac{\Box_{\mathbb{H}}}{m^2}$, where $\Box_{\mathbb{H}}$ is the Laplacian on the unit hyperboloid.

It is standard result that $\Box_{\mathbb{H}}$ is a selfadjoint operator on $\left\{ f \in L^2(\mathbb{H}_m), \, |\Box_{\mathbb{H}} f \in L^2(\mathbb{H}_m) \right\}$ with a continuous negative spectrum; furthermore it commutes with the quasi-regular $O(3,1)$ representation i.e. $[U(\Lambda), \Box_{\mathbb{H}}] = [U(\Lambda), \widetilde{\Box}] = 0$ for any $\Lambda \in O(3,1)$.

The strategy is to consider the mass hyperboloid as a non characteristic initial surface for the wave equation $\widetilde{\Box} u(m, r, \zeta, \epsilon) = 0$ to be solved in the region $m \geq 0$. In particular the following lemma holds:

Lemma 2.2. *Calling $B = -\Box_{\mathbb{H}} - 1$, then for any $f, g \in L^2(\mathbb{H}_m)$ the function*

$$u(m, r, \zeta, \epsilon) = m^{-1+i\sqrt{B}} f(r, \zeta, \epsilon) + m^{-1-i\sqrt{B}} g(r, \zeta, \epsilon) \tag{13}$$

satisfies $\widetilde{\Box} u = 0$ for $m > 0$ with Cauchy data

$$u(1, r, \zeta, \epsilon) = f(r, \zeta, \epsilon) + g(r, \zeta, \epsilon), \quad iB^{-\frac{1}{2}} \frac{\partial(mu)}{\partial m}(1, r, \zeta, \epsilon) = g(r, \zeta, \epsilon) - f(r, \zeta, \epsilon).$$

Furthermore for all $m > 0$ it holds

$$2\left(||f||_2^2 + ||g||_2^2 \right) = \int\limits_{\mathbb{H}_m} d\mu(\mathbb{H}_m) m^2 \left(|u(m, r, \zeta, \epsilon)|^2 + \left| B^{-\frac{1}{2}} \frac{\partial(mu)}{\partial m}(r, \zeta, \epsilon) \right|^2 \right),$$

where $||, ||_2$ is the norm (6) with $\alpha = 2$.

Proof. If we show that $u(m, r, \zeta, \epsilon)$ is a solution of D'Alambert wave equation then the statement on Cauchy data holds per direct substitution and the identity between norms stands per unitarity of the operator $m^{i\sqrt{B}}$ on $L^2(\mathbb{H}_m)$.

Hence let us consider any but fixed $v \in C_0^\infty(\mathbb{R}^4)$ whose support does not include the origin. Dropping the ϵ dependence which is irrelevant to the proof, integration per parts grants:

$$\int_{\mathbb{R}^4} d^4p \; v(p^\mu) \widetilde{\Box} u(p^\mu) = \int_{\mathbb{R}^4} d^4p \; u(p^\mu) \widetilde{\Box} v(p^\mu).$$

In terms of coordinates (m, r, ζ) this last identity reads

$$\int_{\mathbb{R}^4} dm \; d\mu(\mathbb{H}_m) \; m^3 u(m, r, \zeta) \left(-\frac{\partial^2}{\partial m^2} - \frac{3}{m} \frac{\partial}{\partial m} + \frac{\Box_{\mathbb{H}}}{m^2} \right) v(m, r, \zeta) =$$

$$\int_{\mathbb{R}^4} dm \; d\mu(\mathbb{H}_m) \; mv(m, r, \zeta) \left(\Box_{\mathbb{H}} - m^2 \frac{\partial^2}{\partial m^2} - 3m \frac{\partial}{\partial m} \right) u(m, r, \zeta),$$

which, inserting the expression for $u(m, r, \zeta)$ in the hypothesis, becomes

$$\int_{\mathbb{R}^4} dm \; d\mu(\mathbb{H}_m) \; mv(m, r, \zeta) \left(\Box_{\mathbb{H}} + 1 + B \right) u(m, r, \zeta) = 0,$$

being $B \doteq -\Box_{\mathbb{H}} - 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Box$

The choice of the initial surface as the unitary hyperboloid is pure convenience and no generality is lost in this process since it is possible to pick any $\mathbb{H}_m$ and none of the forthcoming results would be modified. The independence from m in the norm identity in the last lemma and the equality $\lim_{m \to 0} m^2 d\mu(\mathbb{H}_m) = d\mu(\mathbb{C})$ suggests that we are now in position to construct a unitary intertwining operator $\widetilde{T} : L^2(\mathbb{C}) \to L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$. As a matter of fact all the needed ingredients can be found in the previous lemma and in formula (12):

Proposition 2.2. *Given any function $F \in L^2(\mathbb{C})$, let us decompose it as $F = F_+ + F_-$ where $+$ represents the contribution of the integral in the ρ-variable between 0 and infinity in (12) whereas the pedex $-$ refers to that between minus infinity and 0. Then, if $f \doteq F_+|_{\mathbb{H}_m}$ and $g \doteq F_-|_{\mathbb{H}_m}$, the function u constructed as in lemma 2.2 coincides with F. Furthermore the map from $F|_{\mathbb{C}} \longrightarrow L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$ is an intertwiner between the $O(3,1)$ representations.*

The demonstration is left to [Stri73].

Remark 2.1. A consequence of the above proposition is that any $f \in L^2(\mathbb{H}_m)$ can be decomposed as

$$f(p^\mu) = \sum_{\epsilon = \pm 1} \int_0^\infty \frac{d\rho}{2\pi^3} \; \rho^2 \int_{\mathbb{S}^2} d\zeta' |\vec{p} \cdot \vec{\zeta}' - \epsilon E|^{-1+i\rho} \left[\psi_0(\rho, \zeta', \epsilon) + sgn\left(\vec{p} \cdot \vec{\zeta}' - \epsilon p_0 \right) \psi_1(\rho, \zeta', \epsilon) \right], \quad (14)$$

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where
$$\psi_0(\rho,\zeta',\epsilon) = \int\limits_{\mathbb{R}^4} d^4p \, \delta(\eta^{\mu\nu}p_\mu p_\nu - m^2)f(p^\mu)|\vec{p}\cdot\vec{\zeta'} - \epsilon p_0|^{-1-i\rho},$$

and
$$\psi_1(\rho,\zeta',\epsilon) = \int\limits_{\mathbb{R}^4} d^4p \, \delta(\eta^{\mu\nu}p_\mu p_\nu - m^2)f(p^\mu)|\vec{p}\cdot\vec{\zeta'} - \epsilon E|^{-1-i\rho}sgn\left(|\vec{p}\cdot\vec{\zeta'} - \epsilon p_0|\right).$$

Let us pinpoint

1. although (14) is written in terms of the global coordinates, we can switch to intrinsic coordinates (r,ζ,ϵ) over $\mathbb{H}_m$ simply substituting $\vec{p}$ with $\vec{\zeta}$ and p_0 with ϵ. In other words we have decomposed a generic function $f \in L^2(\mathbb{H}_m)$ into a direct integral in terms of irreducible representations of $O(3,1)$.

2. proposition 2.2 provides a way to explicitly construct the inverse intertwiner $T = \widetilde{T}^{-1}$: $L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m) \to L^2(\mathcal{C})$. As a matter of fact starting from any two functions $f, g \in L^2(\mathbb{H}_m)$, one can generate a solution of D'Alambert wave equation out of (13) whose restriction to the light cone is a function $F \in L^2(\mathcal{C})$; in a few words $T(f,g) = F$.

   From our perspective this a slightly inconvenient situation since we start with a solution of (5) and hence with a single function $f \in L^2(\mathbb{H}_m)$. Unfortunately the Cauchy problem, upon which (13) is based, requires two initial condition. Hence we adopt the choice to imbed $L^2(\mathbb{H}_m)$ into the diagonal component of $L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$, namely we fix the map $i : L^2(\mathbb{H}_m) \to L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$ such that $i(f) = (f,f)$. Clearly this choice is not unique and the resulting function on the light cone we will construct depends also upon the choice of i.

   To summarise we have set the map $T \circ i : L^2(\mathbb{H}_m) \to L^2(\mathcal{C})$ such that $T(i(f)) = F$.

In order to complete our task, a last question must be answered namely if, to any element of $L^2(\mathcal{C})$, it corresponds a function in Minkowski spacetime which solves the wave equation. A positive answer has been already given in [Stri77] and, thus, we end up with:

Proposition 2.3. If $F(r,\zeta,\epsilon) \in L^2(\mathcal{C})$, then it is the restriction on the light cone of the Fourier transform of a function $\psi \in L^4(M^4, d^4x)$ which solves the wave equation $\Box_\eta \psi(x^\mu) = 0$ with Cauchy data
$$\psi(0,x^i) = f_1(x^i), \quad \frac{\partial\psi}{\partial t}\left(0,x^i\right) = f_2(x^i),$$

with $K^{\frac{1}{2}}f_1(x^i)$ and $K^{-\frac{1}{2}}f_2(x^i) \in L^2(\mathbb{R}^3, d^3x)$ (j=1,2) where $K = \sqrt{-\triangle}$ and $\triangle = \sum\limits_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$.

Furthermore it exists a suitable constant C such that

$$||\psi(x^\mu)||_{L^4(M^4)} \leq C\left(||K^{\frac{1}{2}}f_1(x^i)||_{L^2(\mathbb{R}^3)} + ||K^{-\frac{1}{2}}f_2(x^i)||_{L^2(\mathbb{R}^3)}\right). \tag{15}$$

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Proof. The first part of the proposition is proved in lemma 1 of Strichartz seminal paper [Stri77]. Hence we know that $\phi(x^\mu)$ is a solution for the wave equation lying in $L^4(M^4, d^4x)$ and we need only to focus on Cauchy data. In a standard Minkowski frame with coordinates $x^\mu = (t, \vec{x}) \in \mathbb{R}^4$ we can decompose the solution for the wave equation constructed out of F as

$$\psi(t, \vec{x}) = \int\limits_{\mathbb{R}^3} \frac{d^3p}{\sqrt{16\pi^3|\vec{p}|}} \left[e^{i(\vec{p}\cdot\vec{x} - t|\vec{p}|)} F_+(p) + e^{i(\vec{p}\cdot\vec{x} + t|\vec{p}|)} F_-(p) \right],$$

where F_+ and F_- are respectively the restriction of F to the upper and lower light cone. Taking into account the identity

$$\psi(t, \vec{x}) = -i \int\limits_{\mathbb{R}^3} \frac{d^3p}{\sqrt{16\pi^3}} K^{-\frac{1}{2}} \left[e^{i(\vec{p}\cdot\vec{x} - t|\vec{p}|)} \frac{F_+(p)}{\sqrt{|\vec{p}|}} + e^{i(\vec{p}\cdot\vec{x} + t|\vec{p}|)} \frac{F_-(p)}{\sqrt{|\vec{p}|}} \right],$$

and evaluating this expression for $t = 0$ we discover that $K^{\frac{1}{2}}\psi(0, \vec{x})$ is up to a multiplicative constant complex number the sum of the Fourier transform of $\frac{F_\pm(p)}{\sqrt{2|\vec{p}|}}$; hence, being $F \in L^2(\mathbb{C})$, per Plancherel theorem $K^{\frac{1}{2}}\psi(0, \vec{x}) \in L^2(\mathbb{R}^3, d^3x)$.

Deriving now once in the time variable and exploiting the same kind of identity, we end up with

$$\frac{\partial \psi}{\partial t}(t, \vec{x}) = i \int\limits_{\mathbb{R}^3} \frac{d^3p}{\sqrt{16\pi^3}} K^{\frac{1}{2}} \left[e^{i(\vec{p}\cdot\vec{x} - t|\vec{p}|)} \frac{F_+(p)}{\sqrt{|\vec{p}|}} - e^{i(\vec{p}\cdot\vec{x} + t|\vec{p}|)} \frac{F_-(p)}{\sqrt{|\vec{p}|}} \right].$$

Hence, evaluating at $t = 0$ this expression and still exploiting the Plancherel theorem as in the previous case, we end up with $K^{\frac{1}{2}} \frac{\partial \psi}{\partial t}(0, \vec{x}) \in L^2(\mathbb{R}^3, d^3x)$.

To conclude the demonstration it suffices to notice that the field $\psi(x^\mu)$ and the functions $f_1(x^i), f_2(x^i)$ satisfy the hypotheses of corollary 2 in [Stri77] where the norm estimates (15) for the homogeneous D'Alambert wave equation have been proved. □

Remark 2.2. On an operative ground the solution of the D'Alambert wave equation can be constructed starting from any but fixed $f \in L^2(\mathbb{H}_m)$, map in $F = T(i(f)) \in L^2(\mathbb{C})$, decompose it as in (12) and eventually perform an inverse Fourier transform *i.e*

$$\psi(x^\mu) = \int\limits_{\mathbb{M}^4} \frac{d^4x}{4\pi^2} e^{i\eta^{\mu\nu} x_\mu p_\nu} \sum_{\epsilon = \pm 1} \int\limits_{-\infty}^{\infty} \frac{d\rho}{2\pi^3} \rho^2 \int\limits_{\mathbb{S}^2} d\zeta' |\vec{p}\cdot\vec{\zeta}' - \epsilon p_0|^{-1+i\rho} \left[\psi_0(\rho, \zeta', \epsilon) + \right.$$
$$\left. + sgn\left(\vec{p}\cdot\vec{\zeta}' - \epsilon p_0\right) \psi_1(\rho, \zeta', \epsilon) \right]. \tag{16}$$

2.1.1. *From bulk to null infinity.*

The results from the previous section can be applied to introduce a "projection" of finite-norm solutions ϕ for the massive Klein-Gordon equation to null infinity. In particular let us summarise that all the informations of ϕ can be encoded in the following triplet of data:

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1. the function $\psi(x^\mu)$ constructed as in (16) which solves the massless Klein-Gordon equation of motion along the lines of proposition 2.3,

2. the quasi-regular representation $U'(\Lambda)$,

3. the intertwiner $\widetilde{T} : L^2(\mathcal{C}) \to L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$.

Thus the overall problem reduces to find a projection for $\psi(x^\mu)$ to null infinity.

As a first step let us remember that Minkowski spacetime can be compactified in the Einstein static universe [Wa84]. More in detail, let us consider the coordinates (u, v, θ, φ) being (θ, φ) the standard coordinates on $\mathbb{S}^2$, $u = t + r$ and $v = t - r$ with r as radial coordinate and let us choose as conformal factor

$$\Omega^2 = 4 \left[(1 + u^2)(1 + v^2) \right]^{-1}. \tag{17}$$

Hence the flat metric is rescaled to

$$ds'^2 = \widehat{g}^{\mu\nu} dx_\mu dx_\nu = \frac{4}{(1 + u^2)(1 + v^2)} \left[-dudv + \frac{(u - v)^2}{4} d\mathbb{S}^2(\theta, \varphi) \right],$$

with $d\mathbb{S}^2(\theta, \varphi) \doteq d\theta^2 + \sin^2 \theta d\varphi^2$. If we perform the change of variables

$$T = \tan^{-1} u + \tan^{-1} v, \quad R = \tan^{-1} u - \tan^{-1} v, \tag{18}$$

then we can realize the original Minkowski spacetime as the locus $(-\pi, \pi) \times (-\pi, \pi) \times S^2 \subset \mathbb{R} \times S^3$ with respect to the metric

$$ds'^2 = \widehat{g}^{\mu\nu} dx_\mu dx_\nu = -dT^2 + dR^2 + \sin^2 R \, d\mathbb{S}^2(\theta, \varphi), \tag{19}$$

i.e. that of Einstein static universe $\widehat{M}$. Let us notice that, the closure of the image of Minkowski spacetime in $(\mathbb{R} \times \mathbb{S}^3, \widehat{g}_{\mu\nu})$ is compact and that $\Im^+$ is nothing but the locus $T + R = \pi$.

More importantly this new background in still globally hyperbolic and, if we introduce $\widetilde{\psi} \doteq \Omega^{-1}\psi$, then it is a solution of the Klein-Gordon equation $\Box_{\widehat{g}}\phi - \frac{\widehat{R}}{6}\phi = 0$ where $\Box_{\widehat{g}} \doteq \widehat{g}^{\mu\nu}\widehat{\nabla}_\mu\widehat{\nabla}_\nu$ is the wave operator with respect to the metric $\widehat{g}_{\mu\nu}$ and $\widehat{R} = 1$ is the scalar curvature of Einstein static universe. Furthermore, since the original Cauchy surface $\mathbb{R}^3$ at $t = 0$ is mapped into $T = 0$ in $\widehat{M}$ we can recast the Cauchy problem in proposition 2.3 as

$$\begin{cases} \Box_{\widehat{g}}\widetilde{\psi}(X^\mu) = \frac{\widetilde{\psi}(X^\mu)}{6} \\ \widetilde{\psi}(0, X^i) = f_1(X^i) \\ \frac{\partial\widetilde{\psi}}{\partial T}(0, X^i) = f_2(X^i) \end{cases}, \tag{20}$$

where $X^\mu \doteq (T, X^i) = (T, R, \theta, \varphi)$ and where $\widetilde{K}^{\frac{1}{2}} f_1(X^i) \in L^2(\mathbb{S}^3)$ and $\widetilde{K}^{-\frac{1}{2}} f_2(X^i) \in L^2(\mathbb{S}^3)$ being $\widetilde{K}$ the square-root of the Laplace-Beltrami operator out of the spatial component of the metric (19). Here square integrability is meant with respect to the measure $d\mu = \sin^2 R \sin \theta dR d\theta d\varphi$.

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Hence $\widetilde{\psi}(X^\mu)$ satisfies the Klein-Gordon equation with $m^2 = \frac{1}{6}$, it coincides with $\Omega^{-1}\psi$ in the image of Minkowski spacetime in $\widehat{M}$ and furthermore it lies in $L^4(M^4, \sqrt{|\widehat{g}|}d^4X)$ since

$$||\psi(x^\mu)||^4_{L^4} = \int\limits_{\mathbb{R}^4} |\psi(x^\mu)|^4 d^4x = \int\limits_{\mathbb{R}^4} |\widetilde{\psi}(x^\mu)|^4\Omega^4 d^4x = \int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}\int\limits_{\mathbb{S}^2} d^4X\sqrt{|\widehat{g}|}|\widetilde{\psi}(X^\mu)|^4,$$

where in the last equality we exploited the coordinate change (18).

Unfortunately, since our aim is to project $\widetilde{\psi}$ on null infinity, the best available tools to define a function on $\Im^+$ are trace theorems for Sobolev spaces. In order to exploit them the set of solutions for the wave equation we are taking into account is too big and thus we need to consider only more regular solutions for the wave equation.

To understand which is the less restrictive constraint we have to impose, let us gather all the needed ingredients. As a first step we point out that, being Minkowski spacetime an open set of finite volume (either with respect to Lesbegue measure or with respect to $\sqrt{|\widehat{g}|}dTdRd\mathbb{S}^2(\theta, \varphi)$) in Einstein static universe, then Hölder inequality grants us that $L^p(M) \subset L^q(M)$ for all $1 \leq q < p \leq \infty$. This property can be recast at a level of first order Sobolev spaces in $L^p(M)$ i.e. $W^{1,p}(M) \subset W^{1,q}(M)$ for $1 \leq q < p \leq \infty$.

As a second step we aim to exploit proposition 4.3 in [Sho97] according to which, if Ω is a bounded domain in $\mathbb{R}^N$ with a three dimensional C^1-boundary $\partial\Omega$, then it exists a linear trace operator $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ which is continuous and uniquely determined by the boundary value of the functions $u \in C^1(\overline{\Omega})$. Furthermore the kernel of γ is $W_0^{1,p}(\Omega)$ i.e. the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

Our scenario meets all the geometric requirements in the above hypothesis since Minkowski background is a bounded open set in Einstein static universe $\mathbb{R} \times \mathbb{S}^3$ which, in its turn, can be identified as an open set of $\mathbb{R}^5$. Furthermore the boundary of M consists of two smooth null hypersurfaces - future and past null infinity - and thus, taking into account that $\widetilde{\psi}$ lies in $L^4(M, d^4X)$ and hence in $L^p(M, d^4X)$ for all $1 \leq p \leq 4$, we can apply such proposition only to those $\widetilde{\psi} \in W^{1,p}(M)$ still with $1 \leq p \leq 4$.

With this further condition set and with the inclusion relations between the Sobolev spaces as discussed before, we are entitled to introduce the the map $\gamma|_{\Im^+} : W^{1,p}(M^4) \rightarrow L^q(\Im^+)$ where q can be fixed to any value lower or equal to p. Here $\Im^+$ is the locus $(-\pi, \pi) \times \mathbb{S}^2$ and the measure on $\Im^+$ is the Lesbegue one. Hence being $\Im^+$ in this reference frame an open set of $\mathbb{R} \times \mathbb{S}^2$, each function on $L^q(\Im^+)$ can be also read as an element in $L^q(\mathbb{R} \times S^2)$. This property will be exploited in the next section.

Taking into account that, both from a physical point of view and for the analysis in the next section, it is better to work with Hilbert spaces on the boundary we can summarise the previous discussion as:

Proposition 2.4. *Assume that Minkowski spacetime M is conformally embedded as an open set of Einstein static universe $\left(\widehat{M}, \widehat{g}\right)$ with $\widehat{g}$ as in (19). Then, for any solution of the wave equation $\psi \in L^4(\mathbb{R}^4, d^4x)$ the function $\widetilde{\psi} \doteq \Omega^{-1}\psi \in L^4(\widehat{M}, \sqrt{|\widehat{g}|}d^4x)$ - with Ω chosen as in*

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(17) - *solves* (20). *Furthermore, whenever* $\widetilde{\psi} \in W^{1,p}(M)$ *with* $p \leq 4$, *it exists a continuous projection operator* $\gamma|_{\Im^+} : W^{1,p}(M) \to L^q(\Im^+)$ *where we fix* $q = 2$ *if* $2 \leq p \leq 4$ *whereas* $q = 1$ *if* $p = 1$. *The image* Ψ *under* $\gamma|_{\Im^+}$ *of* $\widetilde{\psi}$ *will be referred to as its restriction on future null infinity.*

Remark 2.3. This last proposition partly overlaps the scenarios envisaged in [DMP06, MaNi04] where only solutions ψ to the D'Alambert wave equation with compactly supported initial data where taken into account. As partly discussed in the introduction, in this case, $\psi \in C^\infty(\mathbb{R}^4)$ and accordingly also $\widetilde{\psi} \in C^\infty(M)$ adopting the nomenclature of the previous analysis. Furthermore, the uniqueness of the solution for the Cauchy problem of the Klein-Gordon equation in the Einstein static universe allows to construct a unique function in $\widehat{M}$ coinciding with $\widetilde{\psi}$ if restricted to M. Hence, in this case, restriction to $\Im^+$ simply means the evaluation of the solution on future null infinity.

Remark 2.4. We point out that the additional regularity condition (*i.e.* $\widetilde{\psi} \in W^{1,p}(M)$ on the solutions for the D'Alambert wave equation haw been set in the Einstein static universe because a direct inspection of the previous construction shows that, although, whenever $f \in L^p(\mathbb{R}^4, d^4x)$, $\Omega^{-1}f \in L^p(\mathbb{R}^4, \sqrt{|\widehat{g}|}d^4x)$ for $p \leq 4$, this does not held true for first order Sobolev spaces. In other words $f \in W^{1,p}(\mathbb{R}^4)$, then, exploiting Liebinitz rule, one can realize that, due to the contribution of the derivatives of the conformal factor (17), $\Omega^{-1}f \in L^p(\mathbb{R}^4, \sqrt{|\widehat{g}|}d^4x)$ but not necessary in $W^{1,p}(\mathbb{R}^4, \sqrt{|\widehat{g}|}d^4x)$.

Hence we have achieved our goal since all the information from the original massive field ϕ satisfying (5) has been projected onto null infinity in the triplet (Ψ, U', T) where U' is the quasi-regular $O(3,1)$ representation acting on the massless field and T is the intertwiner constructed in the previous section. Two natural questions arise at this stage:

- What about Poincaré covariance?

- What is the field theoretical meaning that (Ψ, U', T) contains the information of the massive scalar field?

Let us answer to the first and simpler question. Up to now we have considered only the quasi-regular $O(3,1)$ action on the set $L^2(\mathbb{C})$ or $L^2(\mathbb{H}_m)$. If we want to deal instead with Poincaré covariant scalar field theories, a function ϕ satisfying either (5) or D'Alambert wave equation would transform in a momentum frame as

$$\widetilde{U}(\Lambda, a^\mu)\hat{\phi}(p_\mu) = e^{ia^\mu p_\mu}\hat{\phi}(\Lambda^{-1}p_\mu),$$

where the hat symbol stands for the Fourier transform. This identity supplemented with the constraints $\eta^{\rho\sigma}p_\rho p_\sigma \phi(p_\mu) = m^2\phi(p_\mu)$ with $m^2 \geq 0$ and $sgn(p_0) > 0$ is a unitary irreducible representation for the full Poincaré group [BaRa86].

In order to relate the two above points of view, beside the trivial restriction from $O(3,1)$ to $SO(3,1)$, we need only to invoke the induction-reduction theorem (*c.f.* chapter 18 in [BaRa86])

according to which the quasi-regular representation $U(\Lambda)$ on $L^2(\mathbb{H}_m)$ is

a) the $SO(3,1)$ representation induced from the identity representation of $SO(3)$,

b) the restriction of the scalar Poincaré representation to the Lorentz group. At the same time, if we start from $U(\Lambda)$, it induces the unitary and irreducible scalar representation of the full Poincaré group.

A similar reasoning and conclusion holds if we consider $L^2(\mathcal{C})$ with the associated quasi-regular representation $U'(\Lambda)$.

2.2. *Data reconstruction on null infinity.* In this last subsection we face the last and most important question namely in which sense the information from the bulk massive field projected on null infinity out of $(\widetilde{\psi}, U', T)$ can be interpreted from a classical field theory perspective. To this end we shall exploit some recent analysis according to which it is possible to explicitly construct a diffeomorphism invariant field theory on future null infinity. Afterwards our aim will be to show how the above triplet can be interpreted in terms of such a boundary free field theory.

Bearing in mind the notations and the nomenclatures of subsection 1.1, we review some feature of the construction of a Poincaré invariant field theory on $\mathfrak{S}^+$ - thought as a null differentiable manifold[5] - for smooth scalar fields invariant under the $\mathcal{R}$ subgroup of the BMS as discussed in [ArDa03, DMP06, Dap06]. Such problem has been discussed for the full $SO(3,1) \ltimes C^\infty(\mathbb{S}^2)$; hence here we will adapt that analysis to the specific scenario of bulk Minkowski background.

To this end we shall follow two possible roads: the first starts from a massless bulk scalar field and it imposes BMS invariance on the smooth projection of such a field on null infinity whereas the second ignores the bulk and it constructs a scalar free field theory on $\mathfrak{S}^+$ by means of the Mackey-Wigner programme *i.e.* we only exploit the knowledge of the symmetry group.

We stress that the full construction has been developed for a generic asymptotically flat spacetime due to the universality of the boundary structure. Hence, although both the above mentioned approaches have been fully accounted for in [DMP06, Dap06], here we will only review the details adapted to the case of Minkowski bulk spacetime and, thus, Poincaré symmetry group on null infinity leaving an interested reader to the above cited manuscripts for a careful analysis.

Let us thus start from the first part of this programme; in order to construct a meaningful scalar field theory on $\mathfrak{S}^+$ starting from the bulk, we can focus only smooth real solutions ψ for the D'Alambert wave equation. As per remark 2.3 such a bulk field projects to $\Psi \in C^\infty(\mathfrak{S}^+)$. Then, if we wish to define a suitable representation of $\mathcal{R}$ acting on each Ψ, the following proposition holds [DMP06]:

Proposition 2.5. *Let us take Minkowski spacetime $(M^4, \eta_{\mu\nu})$ and an associated compactified spacetime $(\widehat{M}, \widehat{g}_{\mu\nu})$ (not necessarily Einstein static universe) and let us fix an arbitrary gauge factor ω. Then, for any but fixed $\lambda \in \mathbb{R}$ and for any but fixed $g \in \mathcal{R} \subset BMS$, a representation is $A^{(\lambda)}(g) : C^\infty(\mathfrak{S}^+) \to C^\infty(\mathfrak{S}^+)$ such that the map $t \mapsto A^{(\lambda)}(g_t)\Psi = \lim_{\mathfrak{S}^+} (\omega\Omega)^\lambda g_t^*(\widetilde{\psi})$ is smooth*

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[5]More appropriately one should claim that we are constructing a QFT on the equivalence class of triplets $(\mathfrak{S}^+, n^a, h_{ab})$ associated to the bulk Minkowski spacetime.

for every fixed bulk scalar field ψ with smooth projection Ψ on $\Im^+$ and for every but fixed one-parameter subgroup of the bulk Poincaré group. In the Bondi frame $(u, z, \bar{z})$ it reads

$$\left(A^{(\lambda)}(g)\Psi \right)(u', z', \bar{z}') = K_\Lambda^{-\lambda}(z, \bar{z})\Psi(u, z, \bar{z}), \quad \forall g = (\Lambda, \alpha(z, \bar{z})) \in \mathcal{R}$$

where the primed coordinates and $K_\Lambda(z, \bar{z})$ are defined as in (2) and (3).

Since our aim is to deal with unitary and irreducible representations we have to go one step further *i.e.*

Proposition 2.6. *Let us consider the set $\mathcal{S}(\Im^+) \subset C^\infty(\Im^+)$ of real functions Ψ such that Ψ itself and all its derivatives decay faster than any power of $|u|$ when $|u| \to \infty$ and uniformly in $(z, \bar{z})$. Then $\mathcal{S}(\Im^+)$ can be endowed with the strongly non degenerate symplectic form*

$$\sigma(\Psi_1, \Psi_2) = \int\limits_{\mathbb{R} \times \mathbb{S}^2} \left(\Psi_2 \frac{\partial \Psi_1}{\partial u} - \Psi_1 \frac{\partial \Psi_2}{\partial u} \right) du d\mathbb{S}^2(z, \bar{z}),$$

and $(\mathcal{S}(\Im^+), \sigma)$ is invariant only under $A^{(1)}(g)$. Furthermore if we introduce the positive frequency part $\widehat{\Psi}_+$ of $\Psi \in \mathcal{S}(\Im^+)$ as

$$\widehat{\Psi}_+(E, z, \bar{z}) = \int\limits_{\mathbb{R}} \frac{du}{\sqrt{2\pi}} \, e^{iEu} \Psi(u, z, \bar{z}), \quad E \in [0, \infty) \tag{21}$$

we can write $\widehat{\Psi} = \widehat{\Psi}_+ + \overline{\widehat{\Psi}_+}$. If we denote with $\mathcal{S}(\Im^+)_\mathbb{C}$ the complex linear combinations of these functions $\widehat{\Psi}(E, z, \bar{z})$ then

1. *$\mathcal{S}(\Im^+)_\mathbb{C}$ can be closed to Hilbert space $\mathcal{H}$ with respect to the Hermitian inner product*

$$\langle \widehat{\Psi}_1, \widehat{\Psi}_2 \rangle = -i\sigma(\overline{\widehat{\Psi}_1}, \widehat{\Psi}_2).$$

   Furthermore $(\mathcal{H}, \langle, \rangle)$ is unitary isomorphic to $L^2(\mathbb{R} \times \mathbb{S}^2, EdEd\mathbb{S}^2(z, \bar{z}))$

2. *the representation $A^{(1)}(g)$ of $\mathcal{R}$ on $\mathcal{H}$ acts as*

$$\left(A^{(1)}(g)\widehat{\Psi} \right)(E, z, \bar{z}) = e^{iEK_\Lambda(\Lambda^{-1}z, \Lambda^{-1}\bar{z})\alpha(z, \bar{z})} \widehat{\Psi}(EK_\Lambda(\Lambda^{-1}z, \Lambda^{-1}\bar{z}), \Lambda^{-1}z, \Lambda^{-1}\bar{z}), \tag{22}$$

   for any $g = (\Lambda, \alpha(z, \bar{z})) \in \mathcal{R}$ and $A^{(1)}(g)$ is unitary on $\mathcal{H}$.

The proof of this theorem is a recollection with minor modifications of the demonstration of proposition 2.9, 2.12 and 2.14 in [DMP06]. Hence we refer to such paper for an interested reader.

We now state a useful lemma out of this last proposition:

Lemma 2.3. *The projection on $\Im^+$ of each function $\widetilde{\psi}$ constructed as in proposition 2.4 can be unitary mapped into an element of $(\mathcal{H}, \langle, \rangle)$.*

Proof. In proposition 2.4 we projected a function with support on the image of Minkowski spacetime in Einstein static universe to a function $\Psi \in L^2(\Im^+)$ being $\Im^+$, in that specific background, $(-\pi, \pi) \times \mathbb{S}^2$. Since $\mathbb{S}^2$ is compact and $(-\pi, \pi)$ is an open bounded set of $\mathbb{R}$, Ψ can also be read as an element of $L^2(\mathbb{R} \times \mathbb{S}^2)$. We stress that, switching from the Lesbegue measure in $L^2(\Im^+)$ to the natural $SO(3)$-invariant measure on $\mathbb{S}^2$ for $L^2(\mathbb{R} \times \mathbb{S}^2)$ is harmless.

According to Plancherel theorem and to (21) the Fourier transform $\widehat{\Psi} \in L^2(\mathbb{R} \times S^2, EdEd\mathbb{S}^2)$ and, hence, according to proposition 2.6, it can be unitary mapped in $(\mathcal{H}, \langle, \rangle)$. $\qquad\square$

This concludes the first part of our programme though a complete analysis would require the proof that $A^{(1)}$ is irreducible or how it decomposes in irreducible components. The answer to this question will be a byproduct of the Wigner-Mackey analysis that we discuss now.
Such approach calls for the construction of a classical free field theory on a generic manifold only by means of the symmetry group, $\mathcal{R} \subset BMS$ in our case. Although $\mathcal{R}$ is homomorphic to the Poincarè group we cannot simply refer to the standard construction for a covariant field theory in Minkowski background as discussed to quote just one example in chapter 21 of [BaRa86]. On the opposite we need to consider $\mathcal{R}$ as a subgroup of the BMS and hence we shall adapt the analysis in [DMP06] to this simpler scenario.

Referring to this last cited paper for further details, let us introduce the *character* associated to an element of $N \equiv C^\infty(\mathbb{S}^2)$ as a group homomorphism $\chi : N \to U(1)$. Since N can be endowed with a nuclear topology (see theorem 2.1 in [Dap06]) it can be seen as an element of the Gelfand triplet $N \subset L^2(\mathbb{S}^2) \subset N^*$ where N^* is the set of real continuous linear functionals on N (with the induced topology). Hence, as shown in proposition 3.6 in [DMP06], for any character χ it exists a distribution $\beta \in N^*$ such that

$$\chi(\alpha) = e^{i(\beta, \alpha)}, \tag{23}$$

where $(,)$ stands for the pairing between N^* and N.

Such a result can be applied also to the translational subgroup of the Poincarè group on $\Im^+$ provided either that one exploits the inclusion $T^4 \subset C^\infty(\mathbb{S}^2)$ previously discussed either that the dual space of T^4 - namely $\left(T^4\right)^*$ is characterised in the following way [Mc75]: if we construct the *annihilator* of T^4 as

$$\left(T^4\right)^0 = \left\{ \beta \in N^* \mid (\beta, \alpha(z, \bar{z}) = 0, \ \forall \alpha(z, \bar{z}) \in T^4 \right\},$$

$\left(T^4\right)^*$ is (isomorphic to) the quotient $\frac{N^*}{(T^4)^0}$.

Still referring to [DMP06], the Wigner-Mackey approach for the BMS group introduces the intrinsic *covariant* scalar field on null infinity as a map $\psi : N^* \to \mathcal{H}$ which transforms under the unitary representation D of $SO(3,1) \ltimes C^\infty(\mathbb{S}^2)$ as

$$[D(\Lambda, \alpha(z, \bar{z}))\widetilde{\varphi}](\beta) = \chi_\beta(\alpha)\widetilde{\varphi}(\Lambda^{-1}\beta), \quad \forall (\Lambda, \alpha(z, \bar{z})) \in SO(3,1) \ltimes C^\infty(\mathbb{S}^2)$$

where χ_β is a character.

Whenever the bulk spacetime is the Minkowski background and hence we deal with the $\mathcal{R}$ subgroup of the BMS, the above expression translates in

$$\begin{cases} \widetilde{\varphi} : (T^4)^* \to \mathbb{R} \\ [D(\Lambda, \alpha(z,\bar{z}))\widetilde{\varphi}](\beta) = \chi_\beta(\alpha)\widetilde{\varphi}(\Lambda^{-1}\beta) \quad \forall (\Lambda, \alpha(z,\bar{z})) \in \mathcal{R} \end{cases}, \tag{24}$$

where now β must be thought both as a distribution and as a representative for an equivalence class in the coset $\frac{N^*}{(T^4)^0}$.

Remark 2.5. It is important to point out that, in the above discussion, the real difference between a real scalar field on Minkowski background and on null infinity is due to the action of the representation or more properly of the $U(1)$ phase factor.

To be more precise proposition 3.2 in [Dap06] grants us that, being T^4 a subspace of a locally convex topological linear space - namely $C^\infty(\mathbb{S}^2)$, the coset $\frac{N^*}{(T^4)^0}$ is a 4-dimensional space. Thus, if we introduce the set of dual spherical harmonics Y^*_{lm} with $l = 0,1$, $m = -l,...,l$ defined as $(Y^*_{l'm'}, Y_{lm}(z,\bar{z})) = \delta_{ll'}\delta_{mm'}$, then any $\beta \in (T^4)^*$ can be decomposed as

$$\beta = \sum_{l=0}^{1} \sum_{m=-l}^{l} \beta_{lm} Y^*_{lm}.$$

Hence we can extract from each β the four-vector

$$\beta^\mu = -\sqrt{\frac{3}{4\pi}} (\beta_{00}, \beta_{1-1}, \beta_{10}, \beta_{11}). \tag{25}$$

Moreover we define the action of $\Lambda \in SO(3,1)$ on a generic distribution $\beta \in N^*$ as

$$(\Lambda\beta, \alpha(z,\bar{z})) = (\beta, \Lambda^{-1}\alpha(z,\bar{z})) \quad \forall \alpha(z,\bar{z}) \in C^\infty(\mathbb{S}^2), \tag{26}$$

being the action of Λ on $\alpha(z,\bar{z})$ the one defined in (2) and (3). A direct inspection of proposition 1.1 and of the isomorphism between $\frac{N^*}{(T^4)^0}$ and $(T^4)^*$ shows that β^μ transforms as a covector and the quantity

$$m^2 = \eta_{\mu\nu}\beta^\mu\beta^\nu \tag{27}$$

is $SO(3,1)$ invariant. Furthermore m^2 is also a Casimir for the unitary and irreducible representation of the BMS group and hence also for the $\mathcal{R}$ subgroup. Hence this shows that (24) differs from the counterpart in Minkowski background only in the character.

The covariant scalar field (24) does not transform under an irreducible representation of the $\mathcal{R}$ group and, hence, in a physical language it represents only a kinematically allowed configuration.

On the opposite, if we look for a genuine free field, $\widetilde{\varphi}$ should transform under a unitary and irreducible representation; to overcome such a discrepancy we can still exploit Wigner-Mackey

theory which calls for imposing a further constraint to (24). From a more common perspective in classical field theory this operation amounts to impose on $\widetilde{\varphi}$ the equations of motion written in the momenta representation; for the above scalar field it reads [DMP06]:

$$\left[\eta^{\mu\nu}\beta_\mu\beta_\nu - m^2\right]\widetilde{\varphi}[\beta] = 0, \tag{28}$$

where β_μ is the four vector as in (25).

Two comments on (28) are in due course:

1. the equation under analysis could be recast in the more appropriate language of white noise calculus. In the general framework of BMS free field theory $\widetilde{\varphi}[\beta]$ is a functional over a distribution space which is square integrable with respect to a suitable Gaussian measure μ. Hence (28) should be recast in this scenario in terms of (multiplication) operators acting on $L^2(N^*, d\mu)$ and such analysis has been carried out in [Dap06]. In this paper we can avoid such techniques exploiting the identification of $(T^4)^*$ with $\mathbb{R}^4$ which grants us that (28) acquires the standard meaning *i.e.* the support of $\widetilde{\varphi}[\beta]$ is localised over the mass hyperboloid if $m^2 \neq 0$ and over the light cone if $m^2 = 0$. Most importantly the function $\widetilde{\varphi}$ corresponds to an element in $L^2(\mathcal{C})$.

2. the equations (24) and (28) are equivalent to a function transforming under a unitary and irreducible representation of the Poincaré group induced from the $SO(3)$ or from the $SO(2) \ltimes T^2$ little groups depending if $m \neq 0$ or $m = 0$. At the same time a direct inspection of the analysis of chapter 3 in [DMP06] immediately shows that the representation in (24) is nothing but the scalar BMS representation restricted to the $\mathcal{R}$ subgroup.

Before concluding our analysis we still need the last ingredient which relates the two above constructions of a massless scalar field theory on $\Im^+$.

Theorem 2.1. *A field Ψ on $\Im^+$ constructed as in proposition 2.4 corresponds to a $\mathcal{R}$ field (24) which satisfies (28) with $m = 0$. Hence the representation $A^{(1)}(g)$ is also irreducible on $L^2(\mathbb{R} \times \mathbb{S}^2, EdEd\mathbb{S}^2)$.*

Proof. We provide here a much shorter proof than that of theorem 3.35 in [DMP06]. Let us recall that, according to lemma 2.3 Ψ satisfies (22).

Furthermore, following the characterisation of a light cone imbedded in $\mathbb{R}^4$ as discussed at the beginning of section 2.1 and identifying E with $r \doteq |\vec{p}|$ we end up with $\widehat{\Psi} \in L^2(\mathcal{C})$ being $\widehat{\Psi}$ the Fourier transform of Ψ constructed as in proposition 2.4. According to theorem 1 in [Stri77], $\widehat{\Psi}$ can be read on its own as the restriction on $\mathcal{C}$ of the Fourier transform for a function $\widetilde{\Psi}$ satisfying D'Alambert wave equation and, hence, lying in $L^4(\mathbb{R}^4, d^4x)$. The Fourier transform for $\widehat{\widetilde{\Psi}} \in \mathcal{S}'(\mathbb{R}^4)$ satisfies the constraint $\eta^{\mu\nu}p_\mu p_\nu \widehat{\widetilde{\Psi}} = 0$ and the Poincaré group $\mathcal{R}$ still acts as $A^{(1)}(g)$.

To conclude the demonstration, let us now consider (24) which satisfies (28) with $m = 0$. Exploiting the identification between the distribution β and the covector p_μ, a direct inspection

shows that the scalar $\mathcal{R}$ representation acts on (24) as the representation $A^{(1)}(g)$. Thus each Ψ constructed in proposition 2.4 has been mapped into a massless $\mathcal{R}$ scalar free field. Irreducibility of $A^{(1)}(g)$ is now a consequence of Mackey construction which grants us that the scalar $\mathcal{R}$ (and, thus, the $A^{(1)}(g)$) representation induced from the scalar $E(2)$ representation is irreducible. $\quad\square$

We have now all ingredients to conclude our analysis on the projection of a massive bulk scalar field:

Theorem 2.2. *Let us consider any norm-finite solution ϕ of* (5) *with the associated triplet* $(\psi, U', \widetilde{T})$. *The latter projects to a triple* $(\Psi, U', \widetilde{T})$ *on future null infinity which identifies two Poincaré invariant free scalar field constructed à la Wigner-Mackey and solving* (28) *with the same mass value as ϕ.*

Proof. According to the hypothesis of the theorem we can associate to ϕ the triplet $(\psi, U', \widetilde{T})$ where ψ can be written as (16). We can now exploit proposition 2.4 to project ψ in a square integrable function Ψ over $\Im^+$: $\Psi = \rho(\widetilde{\psi})$ where $\widetilde{\psi} \doteq \Omega^{-1}\psi$. Hence, being U' and $\widetilde{T}$ respectively a representation and an intertwiner thus independent from coordinates, we construct on null infinity the triplet $(\Psi, U', \widetilde{T})$. The representation $U'(\Lambda)$ is the quasi-regular representation of the Lorentz group and it unambiguously induces (or it is the restriction of) the scalar $\mathcal{R}$ representation which acts on Ψ as the representation $A^{(1)}(g)$ from (22). We can now exploit theorem 2.1 according to which the pair $(\Psi, A^{(1)}(g))$ corresponds to one $\mathcal{R}$ invariant field (24) which satisfies (28) with $m = 0$. Hence $(\Psi, A^{(1)}(g))$ can be traded with $(\widetilde{\varphi}, D(\Lambda, \alpha(z, \bar{z}))$ where $(\Lambda, \alpha(z, \bar{z})) \in \mathcal{R}$ and D is the scalar representation in (24).

Still the induction-reduction theorem for group representation (chapter 18 in [BaRa86]) grants us that the restriction of D to $SO(3, 1)$ is exactly $U'(g)$ and that the quasi-regular representation unambiguously induces the scalar $\mathcal{R}$ representation. Hence we have mapped the original triplet $(\Psi, U', \widetilde{T})$ in $(\widetilde{\varphi}, U', \widetilde{T})$. The circle has been almost closed and our last step consists of exploiting the same reasoning as in the proof of theorem 2.1 *i.e.* we can read $\widetilde{\varphi}$ as a solution from the massless wave equation constructed out of an element of $L^2(\mathcal{C})$ - say $\hat{\widetilde{\varphi}}|_{\mathcal{C}}$. Hence we can now exploit our last ingredient namely the intertwiner $\widetilde{T} : L^2(\mathcal{C}) \to L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$ *i.e.* $\widetilde{T}(\hat{\widetilde{\varphi}}|_{\mathcal{C}}) = (f, g)$. Accordingly both f and g lies in $L^2(\mathbb{H}_m)$ and the Lorentz group acts as $[U'(\Lambda)f](p^\mu) = f(\Lambda^{-1}p^\mu)$ for all $\Lambda \in SO(3, 1)$. Still exploiting theorem 1 in [Stri77], we can interpret f (or g) as the restriction on the mass hyperboloid $\mathbb{H}_m$ of a function - say $\widetilde{\varphi}_f$ or $\widetilde{\varphi}_g$ - whose Fourier transform satisfies the Klein-Gordon equation of motion with mass m and it lies in $L^p(\mathbb{R}^4, d^4x)$ with $\frac{10}{3} \le p \le 4$. If we now take into account that the original field $\widetilde{\varphi}$ is an intrinsic $\mathcal{R}$ free field, we are entitled to switch from p^μ to the variables β^μ. To conclude we can exploit remark 2.5 according to which a covector β^μ transforming under the standard $SO(3, 1)$ action corresponds to a distribution $\beta \in (T^4)^* \subset N^*$ on which $\Lambda \in SO(3, 1)$ acts according to (26). Eventually still the induction theorem allow us to construct from $U'(\Lambda)$ the scalar $\mathcal{R}$ representation D. Hence both $(\widetilde{\varphi}_f, U'(g))$ and $(\widetilde{\varphi}_g, U'(g))$ correspond unambiguously to a $\mathcal{R}$ massive scalar field as in (24) with support on the mass hyperboloid *i.e.* with the same value for m^2 as the original Minkowski field ϕ.

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$\square$

Remark 2.6.   The projection of a bulk massive scalar field into two boundary massive scalar fields is a natural byproduct of the intertwining operator. In the projection of $f \in L^2(\mathbb{H}_m)$ to a function over the light cone, we could imbed f into the element (f, f) of the diagonal subgroup of $L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$; on the opposite on the boundary we perform the inverse operation mapping a square integrable function over the light cone into $L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$. Hence there is no guarantee that the intertwiner identifies an element of the diagonal subgroup and we are forced to take into account two massive fields instead of a single one.

3   Conclusions

In this paper we have shown that, exploiting Strichartz harmonic analysis on hyperboloids, it is possible to project the information of a norm finite massive real scalar field ϕ in Minkowski spacetime into null a triplet of data on null infinity: $(\Psi, U', \widetilde{T})$ where Ψ is the projection on $\Im^+$ out of trace operator of a solution for the D'Alambert wave equation, U' is the $SO(3,1)$ quasi-regular representation whereas $\widetilde{T}$ is a unitary intertwiner from $L^2(\mathbb{C})$ to $L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$.

The result we achieve has a twofold advantage. From one side it is coherent with Helfer result which states that the space of section for any vector bundle over null infinity carries only the massless representation for the homogeneous action of the Poincaré group. As a matter of fact Ψ can be ultimately interpreted as a free field on the conformal boundary with $m = 0$. From the other side we can recover the original interpretation of massive fields exploiting the action of $\widetilde{T}$ and, as shown in theorem 2.2, the original single field ϕ corresponds to two separate massive free fields in the $\mathcal{R}$ invariant theory constructed à la Wigner-Mackey.

Although we believe the result is rather appealing opening a wide range of possible applications, it is fair to admit that it is to a certain extent not sharp. As a matter of fact, in the whole construction we performed three arbitrary choices: the first, already discussed, refers to the imbedding of an element $f \in L^2(\mathbb{H}_m)$ (the restriction on the mass hyperboloid of the Fourier transform of ϕ) into the diagonal subgroup of $L^2(\mathbb{H}_m) \oplus L^2(\mathbb{H}_m)$.

The second and the strongest between the performed choices arises in the projection to $\Im^+$; the general solution $\widetilde{\psi}$ of the D'Alambert wave equation we constructed lies in $L^4(M^4, \sqrt{|g|}d^4x)$ but, in order to apply trace theorems, we needed to consider at least Sobolev spaces of first order. This restricts the range of validity of our results and it will be interesting to eliminate such constraint from our analysis.

The third and less pernicious of the choices lies in the construction of the above mentioned trace operator. As a matter of fact we embedded Minkowski spacetime into an open region of Einstein static universe. Hence this amount to select a preferred gauge factor ω according to the definitions of section 1.1 contrary to the projection operator introduced in [DMP06, MaNi04] which provides a smooth function over $\Im^+$ for any possible choice of ω. Nonetheless we feel that, fixing ω in our analysis, does not lead to a loss of generality since we can ultimately interpret our results in terms of a general field theory constructed over $\Im^+$ without the need for a choice for the gauge factor.

To conclude we wish to discuss possible applications of our results. Our main target is an holographic interpretation of bulk field theory along the lines of [DMP06] and the previous section was written with this goal in mind. As a matter of fact we have proved that, at least in Minkowski background, it is possible to project each solution of a massive Klein-Gordon equation of motion into a suitable counterpart at null infinity. Such bulk-to-boundary interplay does not represent the only possible application of our analysis and we envisage that our results could be possibly exploited for other research fields such as, to quote an example, conformal scattering problems.

Nonetheless we believe that the most interesting perspective consists of the development of a similar result for generic globally hyperbolic and asymptotically flat spacetime. Already at a first reading of this manuscript, one can realize that the extensive use of tools proper of harmonic analysis forbids to mimic our procedure in a more generic scenario[6]. Nonetheless we feel that finding a way to project the information for a massive scalar field on null infinity in Minkowski background is an encouraging starting point to deal with the same problem in more complicated frameworks. A positive conclusion of such a research project would also open the way to develop in a generic background (and not only in Minkowski), with tools proper of the algebraic quantisation of field theory, a correspondence between the quantised bulk massive scalar theory and the bulk counterpart. We aim to address such an issue in future works.

Acknowledgments

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[6]It is possible to wonder if our construction extends to higher dimensional flat backgrounds and, since both the tools proper of harmonic analysis and Strichartz estimates can be generalised to any dimension, there is no apparent obstruction. On the opposite the question if one can derive a similar result for curved backgrounds is less adequate since conformal completion techniques à la Penrose may run into serious difficulties [HoIs05].

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